BOUNDEDNESS OF THE IMAGES OF PERIOD MAPS

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Abstract. We prove a conjecture of Griffiths on simultaneous normalization of all periods which asserts that the image of the lifted period map on the universal cover lies in a bounded domain in complex Euclidean space.

Introduction

First we introduce the notations in this paper. Following the terminology of Griffiths, all algebraic varieties are assumed to be smooth over $\mathbb{C}$ and morphisms between algebraic varieties to be rational and holomorphic. In this paper, we will consider the variation of Hodge structure arising from geometry. This means that we will consider an algebraic family of polarized algebraic varieties, which is a proper morphism between algebraic varieties $f : \mathfrak{X} \to S$ with the following properties

(1) the varieties $\mathfrak{X}$ and S are smooth and connected, and the morphism f is nondegenerate, i.e. the tangent map df is of maximal rank at each point of $\mathfrak{X}$;
(2) $\mathfrak{X} \subseteq \mathbb{P}^N$ is quasi-projective with restricted polarization $\mathcal{L}$ over $\mathfrak{X}$;
(3) each fiber $X_s = f^{-1}(s)$, $s \in S$, is smooth, connected, and projective with the polarization $L_s = \mathcal{L}|_{X_s}$.

In general, S is not compact, but it admits a smooth compactification due to Hironaka, i.e. we can embed S as a Zariski open subset of a complete smooth algebraic variety $\overline{S}$ such that $\overline{S} - S$ is divisor with simple normal crossings. That is to say that $\overline{S} - S$ is locally defined by the equation $z_1 \cdots z_k = 0$ in suitable local coordinates $\{z_1, \cdots, z_n\}$ with $k \leq \dim_{\mathbb{C}} S = n$.

Since each fiber $X = X_s$ is projective with polarization $L = \mathcal{L}|_X \in H^2(X, \mathbb{Z})$, we can define the primitive cohomology group by

$$H^n_{pr}(X, \mathbb{F}) = \ker(L : H^n(X, \mathbb{F}) \to H^{n+2}(X, \mathbb{F})),$$

where $n = \dim_{\mathbb{C}} X$ and $\mathbb{F}$ can be $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{C}$. From Hodge theory we have the Hodge decomposition of the primitive cohomology group

$$H^n_{pr}(X, \mathbb{C}) = H^{n,0}_{pr}(X) \oplus H^{n-1,1}_{pr}(X) \oplus \cdots \oplus H^{0,n}_{pr}(X),$$

where $H^{k,n-k}_{pr}(X) = H^{k,n-k}(X) \cap H^n_{pr}(X, \mathbb{C})$. The Poincaré bilinear form Q on $H^n_{pr}(X, \mathbb{C})$ is defined by

$$Q(u, v) = (-1)^{\frac{n(n-1)}{2}} \int_X u \wedge v$$

for any d-closed primitive n-forms u, v on X. Then Q is nondegenerate and satisfies the Hodge-Riemann relations

$$Q\left(H^{n-k,k}_{pr}(X), H^{n-l,l}_{pr}(X)\right) = 0 \text{ unless } k+l=n; \tag{1}$$

$$\left(\sqrt{-1}\right)^{2k-n} Q\left(v,\overline{v}\right) > 0 \text{ for } v \in H^{k,n-k}_{pr}(X)\setminus\{0\}. \tag{2}$$

Let $h^{k,n-k} = \dim_{\mathbb{C}} H^{k,n-k}_{pr}(X)$ and $f^k = \sum_{i=k}^n h^{i,n-i}$. Define the decreasing filtration $H^n_{pr}(X,\mathbb{C}) = F^0 \supset \cdots \supset F^n = 0$ by taking $F^k = F^k(X) = H^{n,0}_{pr}(X) \oplus \cdots \oplus H^{k,n-k}_{pr}(X)$. We know that

$$\begin{aligned} &\dim_{\mathbb{C}} F^k = f^k, \\ &H^n_{pr}(X,\mathbb{C}) = F^k \oplus \overline{F^{n-k+1}}, \text{ and } H^{k,n-k}_{pr}(X) = F^k \cap \overline{F^{n-k}}. \end{aligned} \tag{3}$$

In terms of Hodge filtrations, the Hodge-Riemann relations (1) and (2) are

$$Q\left(F^k, F^{n-k+1}\right) = 0; \tag{4}$$

$$Q\left(Cv, \overline{v}\right) > 0 \text{ if } v \neq 0, \tag{5}$$

where C is the Weil operator given by $Cv = \left(\sqrt{-1}\right)^{2k-n} v$ for $v \in H^{k,n-k}_{pr}(X)$. The period domain D for polarized Hodge structures is the space of all such Hodge filtrations

$$D = \left\{F^n \subset \cdots \subset F^0 = H^n_{pr}(X,\mathbb{C}) \mid (3), (4) \text{ and } (5) \text{ hold}\right\}.$$

The compact dual $\check{D}$ of D is

$$\check{D} = \left\{F^n \subset \cdots \subset F^0 = H^n_{pr}(X,\mathbb{C}) \mid (3) \text{ and } (4) \text{ hold}\right\}.$$

One can prove that $\check{D}$ is an algebraic manifold and the period domain $D \subseteq \check{D}$ is an open submanifold. See Theorem 4.3 in [9] for a complete proof and Proposition 8.2 in [11] for an alternative proof.

From the definition of period domain we naturally get the Hodge bundles on $\check{D}$ by associating to each point in $\check{D}$ the vector spaces $\{F^k\}_{k=0}^n$ in the Hodge filtration of that point. We will denote the Hodge bundles by $\mathscr{F}^k \to \check{D}$ with $\mathscr{F}^k|_p = F^k_p$ as the fiber for any $p \in \check{D}$ and each $0 \leq k \leq n$.

For the family $f : \mathfrak{X} \to S$ and some fixed point $s_0 \in S$, the period map is defined as a morphism $\Phi : S \to D/\Gamma$ by

$$s \mapsto \tau^{[\gamma_s]}(F^n_s \subseteq \cdots \subseteq F^0_s) \in D, \tag{6}$$

where $F^k_s = F^k(X_s)$ and $\tau^{[\gamma_s]}$ is an isomorphism between $\mathbb{C}-$vector spaces

$$\tau^{[\gamma_s]}: H^n(X_s,\mathbb{C}) \to H^n(X_{s_0},\mathbb{C}),$$

which depends only on the homotopy class $[\gamma_s]$ of the curve γ_s between s and s_0. Then the period map is well-defined with respect to the monodromy representation $\rho : \pi_1(S) \to \Gamma \subseteq \mathrm{Aut}(H_{\mathbb{Z}}, Q)$. It is well-known that the period map has the following properties:

(i) locally liftable;
(ii) holomorphic: $\partial F^i_s/\partial \overline{s} \subseteq F^i_s$, $0 \leq i \leq n$;
(iii) Griffiths transversality: $\partial F^i_s/\partial s \subseteq F^{i-1}_s$, $1 \leq i \leq n$.

Thanks to (i) we can lift the period map to $\widetilde{\Phi}:\mathcal{T}\to D$ by taking the universal cover $\mathcal{T}$ of S such that the diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "\widetilde{\Phi}"] \arrow[d, "\pi"] & D \arrow[d, "\pi"] \\ S \arrow[r, "\Phi"] & D/\Gamma \end{tikzcd} \tag{7}$$

is commutative. Without loss of generality we will assume that Γ is torsion free, and therefore D/Γ is smooth. Otherwise, we first choose Γ as the whole $\mathrm{Aut}(H_{\mathbb{Z}},Q)$, and then take a torsion free subgroup of Γ and proceed on a finite cover of S. For the details, one can see the proof of Lemma IV-A, page 705-706 in [38].

In his paper [12], Griffiths raised the following conjecture as Conjecture 10.1 in Section 10, which is now the main theorem of our paper.

Theorem 0.1. (Griffiths Conjecture) *Given $f:\mathfrak{X}\to S$, there exists a simultaneous normalization of all the periods $\Phi(X_s)$ $(s\in S)$. More precisely, the image $\widetilde{\Phi}(\mathcal{T})$ lies in a bounded domain in a complex Euclidean space.*

The main idea of our proof is Riemann extension theorem which asserts that

- Suppose that M is a complex manifold and $V\subseteq M$ is an analytic subvariety with $\mathrm{codim}_{\mathbb{C}}V\geq 1$. Then any holomorphic function f defined on $M\setminus V$, which is locally bounded near V, can be extended uniquely to a global holomorphic function $\widetilde{f}$ on M such that $\widetilde{f}|_{M\setminus V}=f$.

Based on this main idea, our proof can be divided into the following steps:

Step 1 : Find an analytic subvariety of $\mathrm{codim}_{\mathbb{C}}\geq 1$.

To explain the detail we need to review some results of period domain from Lie theory. We fix a point p in $\mathcal{T}$ and its image $o=\widetilde{\Phi}(p)$ as the reference points or base points. If we define the complex Lie group as

$$G_{\mathbb{C}}=\{g\in GL(H_{\mathbb{C}})|\ Q(gu,gv)=Q(u,v)\text{ for all }u,v\in H_{\mathbb{C}}\},$$

then $G_{\mathbb{C}}$ acts transitively on $\check{D}$ with stabilizer $B=\{g\in G_{\mathbb{C}}|gF_p^k=F_p^k,\ 0\leq k\leq n\}$. Let $G_{\mathbb{R}}\subseteq G_{\mathbb{C}}$ be the real subgroup which maps $H_{\mathbb{R}}$ to $H_{\mathbb{R}}$, then we can realize D as $D=G_{\mathbb{R}}/V$ with $V=B\cap G_{\mathbb{R}}$. On the Lie algebra $\mathfrak{g}$ of $G_{\mathbb{C}}$ we can define a Hodge structure of weight zero by

$$\mathfrak{g}=\bigoplus_{k\in\mathbb{Z}}\mathfrak{g}^{k,-k}\quad\text{with}\quad\mathfrak{g}^{k,-k}=\{X\in\mathfrak{g}|XH_p^{r,n-r}\subseteq H_p^{r+k,n-r-k},\ 0\leq r\leq n-k\}.$$

By definition the Lie algebra of B is $\mathfrak{b}=\bigoplus_{k\geq 0}\mathfrak{g}^{k,-k}$ and the holomorphic tangent space of $\check{D}$ at the base point o is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}\simeq\oplus_{k\geq 1}\mathfrak{g}^{-k,k}\triangleq\mathfrak{n}_+$. We denote the unipotent group to be $N_+=\exp(\mathfrak{n}_+)$. Since $N_+\cap B=\{\mathrm{Id}\}$, we can identify the unipotent group $N_+\subseteq G_{\mathbb{C}}$ with its orbit $N_+(o)\subseteq\check{D}$ so that $N_+\subseteq\check{D}$ is meaningful. With this we define $\check{\mathcal{T}}=(\widetilde{\Phi})^{-1}(N_+\cap D)$ and show that $\mathcal{T}\setminus\check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}\setminus\check{\mathcal{T}})\geq 1$.

Step 2 : Show that $\widetilde{\Phi}|_{\check{\mathcal{T}}}:\check{\mathcal{T}}\to N_+\cap D$ is bounded.

Now we study in detail the structure of the Lie algebras involved. Suppose that the Weil operator $\theta : \mathfrak{g} \to \mathfrak{g}$ is defined by

$$\theta(X) = (-1)^k X, \text{ for } X \in \mathfrak{g}^{k,-k}.$$

Then we can decompose Lie algebra $\mathfrak{g}$ into the union of eigenspaces of the Weil operator as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ and $\mathfrak{p}$ correspond to the eigenvalues $+1$ and -1 respectively.

Let $\mathfrak{g}_0 \subseteq \mathfrak{g}$ be the Lie algebra of $G_{\mathbb{R}}$ and $\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0$, $\mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0$, then we get the decomposition of $\mathfrak{g}_0$ as

$$\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0.$$

In fact, Schmid showed that this decomposition is Cartan decomposition, i.e. $\mathfrak{g}_c \triangleq \mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0$ is a compact real semi-simple Lie algebra.

Next we choose a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{h}_0$ is also a Cartan subalgebra of $\mathfrak{k}_0$, where $\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}$ is the Lie algebra of V. In Lie theory, root system plays a central role in the structures of Lie algebras. With the root system, one has the decomposition of the Lie algebra as

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\varphi \in \Delta} \mathfrak{g}^{\varphi}.$$

In our case, we decompose the root system Δ into the union of $\Delta_{\mathfrak{k}}$ and $\Delta_{\mathfrak{p}}$ due to the corresponding space $\mathfrak{g}^{\varphi} \subseteq \mathfrak{k}$ or $\mathfrak{p}$. Then we introduce the notion of strongly orthogonal which says that two different roots $\varphi, \psi \in \Delta$ are strongly orthogonal if and only if $\varphi \pm \psi \notin \Delta \cup \{0\}$. The following two properties are key to our proof.

(1) There exists a set of strongly orthogonal noncompact positive roots

$$\Lambda = \{\varphi_1, \cdots, \varphi_r\} \subseteq \Delta_{\mathfrak{p}}^+$$

such that

$$\mathfrak{A}_0 = \sum_{i=1}^{r} \mathbb{R}\left(e_{\varphi_i} + e_{-\varphi_i}\right)$$

is a maximal abelian subspace in $\mathfrak{A}_0$.

(2) If $\mathfrak{A}_0'$ is any maximal abelian subspace of $\mathfrak{p}_0$, then there exists an element $k \in K$, a maximal compact subgroup of G, such that $\mathrm{Ad}(k) \cdot \mathfrak{A}_0 = \mathfrak{A}_0'$. See Section 2 for the definition of K. Moreover, we have

$$\mathfrak{p}_0 = \bigcup_{k \in K} \mathrm{Ad}(k) \cdot \mathfrak{A}_0$$

Let $\mathfrak{a} \subseteq \mathfrak{n}_+$ be the abelian subalgebra of $\mathfrak{n}_+$ determined by the period map $\widetilde{\Phi}$. Let $A \triangleq \exp(\mathfrak{a}) \subseteq N_+$ which is isomorphic to a complex Euclidean subspace, and $P : N_+ \cap D \to A \cap D$ be the induced projection map. The restricted period map $\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$ composed with the projection map P gives a holomorphic map $\Psi : \check{\mathcal{T}} \to A \cap D$ by $\Psi = P \circ \widetilde{\Phi}$.

In Lemma 3.1 we prove that $\Psi : \check{\mathcal{T}} \to A \cap D$ is bounded with respect to the Euclidean metric on $A \subseteq N_+$. In Theorem 3.3, we prove the boundedness of $\widetilde{\Phi}(\check{\mathcal{T}})$ in N_+ by the

finiteness of the map $P|_{\check{\mathcal{T}}}$, where we essentially use the Griffiths transversality of the extended period map, which is introduced in Section 1.

Step 3 : By Step 1 and Step 2 and the Riemann Extension Theorem, we can finish the proof of Theorem 0.1.

We remark that without further assumptions, we can only prove that the image $\widetilde{\Phi}(\mathcal{T})$ lies in $\mathbb{C}^N$ as a bounded subvariety. In many cases, however, we can embed $\widetilde{\Phi}(\mathcal{T})$ in $\mathbb{C}^N$ as a complex sub-manifold, even as bounded open domain. We will come back to this in our future work.

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1. Period domains from the viewpoint of Lie theory

In this section we review the definitions and basic properties of period domains from Lie theory point of views. We consider the nilpotent Lie subalgebra $\mathfrak{n}_+$ and define the corresponding unipotent group to be $N_+ = \exp(\mathfrak{n}_+)$. Since $N_+ \cap B = \{\mathrm{Id}\}$, we can identify the unipotent group $N_+ \subseteq G_{\mathbb{C}}$ with its orbit $N_+(o) \subseteq \check{D}$. Then we define $\check{\mathcal{T}} = \widetilde{\Phi}^{-1}(N_+ \cap D)$ and show that $\mathcal{T}\backslash\check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}\backslash\check{\mathcal{T}}) \geq 1$.

First we fix a point p in $\mathcal{T}$ and its image $o = \widetilde{\Phi}(p)$ as the reference points or base points. Let us introduce the notion of adapted basis for the given Hodge decomposition or Hodge filtration. For the fixed point $p \in \mathcal{T}$ and $f^k = \dim F_p^k$ for any $0 \leq k \leq n$, we call a basis

$$\xi = \left\{\xi_0, \cdots, \xi_{f^n-1}, \xi_{f^n}, \cdots, \xi_{f^{n-1}-1} \cdots, \xi_{f^{k+1}}, \cdots, \xi_{f^k-1}, \cdots, \xi_{f^1}, \cdots, \xi_{f^0-1}\right\}$$

of $H^n_{pr}(X_p, \mathbb{C})$ an fadapted basis for the given Hodge decomposition

$$H^n_{pr}(X_p, \mathbb{C}) = H_p^{n,0} \oplus H_p^{n-1,1} \oplus \cdots \oplus H_p^{1,n-1} \oplus H_p^{0,n},$$

if it satisfies $H_p^{k,n-k} = \mathrm{Span}_{\mathbb{C}}\left\{\xi_{f^{k+1}}, \cdots, \xi_{f^k-1}\right\}$ with $h^{k,n-k} = f^k - f^{k+1}$. We call a basis

$$\zeta = \left\{\zeta_0, \cdots, \zeta_{f^n-1}, \zeta_{f^n}, \cdots, \zeta_{f^{n-1}-1} \cdots, \zeta_{f^{k+1}}, \cdots, \zeta_{f^k-1}, \cdots, \zeta_{f^1}, \cdots, \zeta_{f^0-1}\right\}$$

of $H^n_{pr}(X_p, \mathbb{C})$ an adapted basis for the given filtration

$$F_p^n \subseteq F_p^{n-1} \subseteq \cdots \subseteq F_p^0$$

if it satisfies $F_p^k = \mathrm{Span}_{\mathbb{C}}\{\zeta_0, \cdots, \zeta_{f^k-1}\}$ with $\dim_{\mathbb{C}} F^k = f^k$. For the convenience of notations, we set $f^{n+1} = 0$ and $m = f^0$.

The blocks of an $m \times m$ matrix T are set as follows: for each $0 \leq \alpha, \beta \leq n$, the (α, β)-th block $T^{\alpha,\beta}$ is

$$T^{\alpha,\beta} = (T_{ij})_{f^{-\alpha+n+1} \leq i \leq f^{-\alpha+n}-1,\ f^{-\beta+n+1} \leq j \leq f^{-\beta+n}-1}, \tag{8}$$

where T_{ij} is the entries of the matrix T. In particular, $T = (T^{\alpha,\beta})$ is called a block lower triangular matrix if $T^{\alpha,\beta} = 0$ whenever $\alpha < \beta$.

Let $H_{\mathbb{F}} = H^n_{pr}(X, \mathbb{F})$, where $\mathbb{F}$ can be chosen as $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$. We define the complex Lie group

$$G_{\mathbb{C}} = \{g \in GL(H_{\mathbb{C}}) |\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{C}}\},$$

and the real one

$$G_{\mathbb{R}} = \{g \in GL(H_{\mathbb{R}})|\ Q(gu, gv) = Q(u, v) \text{ for all } u, v \in H_{\mathbb{R}}\}.$$

Griffiths in [9] showed that $G_{\mathbb{C}}$ acts on $\check{D}$ transitively and so does $G_{\mathbb{R}}$ on D. The stabilizer of $G_{\mathbb{C}}$ on $\check{D}$ at the fixed point o is

$$B = \{g \in G_{\mathbb{C}}| gF_p^k = F_p^k,\ 0 \le k \le n\},$$

and the one of $G_{\mathbb{R}}$ on D is $V = B \cap G_{\mathbb{R}}$. Thus we can realize $\check{D}$ as

$$\check{D} = G_{\mathbb{C}}/B, \text{ and } D = G_{\mathbb{R}}/V$$

so that $\check{D}$ is an algebraic manifold and $D \subseteq \check{D}$ is an open complex submanifold. One can find a complete proof of this result in Theorem 4.3 of [9], or Proposition 8.2 in [11] for an alternative proof.

The Lie algebra $\mathfrak{g}$ of the complex Lie group $G_{\mathbb{C}}$ is

$$\mathfrak{g} = \{X \in \mathrm{End}(H_{\mathbb{C}})|\ Q(Xu, v) + Q(u, Xv) = 0, \text{ for all } u, v \in H_{\mathbb{C}}\},$$

and the real subalgebra

$$\mathfrak{g}_0 = \{X \in \mathfrak{g}|\ XH_{\mathbb{R}} \subseteq H_{\mathbb{R}}\}$$

is the Lie algebra of $G_{\mathbb{R}}$. Note that $\mathfrak{g}$ is a simple complex Lie algebra and contains $\mathfrak{g}_0$ as a real form, i.e. $\mathfrak{g} = \mathfrak{g}_0 \oplus \sqrt{-1}\mathfrak{g}_0$.

On the linear space $\mathrm{Hom}(H_{\mathbb{C}}, H_{\mathbb{C}})$ we can give a Hodge structure of weight zero by

$$\mathfrak{g} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}^{k,-k} \quad \text{with} \quad \mathfrak{g}^{k,-k} = \{X \in \mathfrak{g}| XH_p^{r,n-r} \subseteq H_p^{r+k,n-r-k},\ 0 \le r \le n-k\}.$$

By definition of B the Lie algebra $\mathfrak{b}$ of B has the form $\mathfrak{b} = \bigoplus_{k \ge 0} \mathfrak{g}^{k,-k}$. Then the Lie algebra $\mathfrak{v}_0$ of V is

$$\mathfrak{v}_0 = \mathfrak{g}_0 \cap \mathfrak{b} = \mathfrak{g}_0 \cap \mathfrak{b} \cap \overline{\mathfrak{b}} = \mathfrak{g}_0 \cap \mathfrak{g}^{0,0}.$$

With the above isomorphisms, the holomorphic tangent space of $\check{D}$ at the base point is naturally isomorphic to $\mathfrak{g}/\mathfrak{b}$.

Let us consider the nilpotent Lie subalgebra $\mathfrak{n}_+ := \oplus_{k \ge 1} \mathfrak{g}^{-k,k}$. Then one gets the holomorphic isomorphism $\mathfrak{g}/\mathfrak{b} \cong \mathfrak{n}_+$. We denote the unipotent group to be

$$N_+ = \exp(\mathfrak{n}_+).$$

As $\mathrm{Ad}(g)(\mathfrak{g}^{k,-k})$ is in $\bigoplus_{i \ge k} \mathfrak{g}^{i,-i}$ for each $g \in B$, the subspace $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b} \subseteq \mathfrak{g}/\mathfrak{b}$ defines an $\mathrm{Ad}(B)$-invariant subspace. By left translation via $G_{\mathbb{C}}$, $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$ gives rise to a $G_{\mathbb{C}}$-invariant holomorphic subbundle of the holomorphic tangent bundle. It will be denoted by $T_h^{1,0}\check{D}$, and will be referred to as the horizontal tangent subbundle. One can check that this construction does not depend on the choice of the base point. The horizontal tangent subbundle, restricted to D, determines a subbundle $T_h^{1,0}D$ of the holomorphic tangent bundle $T^{1,0}D$ of D. The $G_{\mathbb{C}}$-invariace of $T_h^{1,0}\check{D}$ implies the $G_{\mathbb{R}}$-invariance of $T_h^{1,0}D$. Note that the horizontal tangent subbundle $T_h^{1,0}D$ can also be constructed as the associated bundle of the principle bundle $V \to G_{\mathbb{R}} \to D$ with the adjoint representation of V on

the space $\mathfrak{b} \oplus \mathfrak{g}^{-1,1}/\mathfrak{b}$. As another interpretation of the horizontal bundle in terms of the Hodge bundles $\mathscr{F}^k \to \check{D}$, $0 \leq k \leq n$, one has

$$T_h^{1,0}\check{D} \simeq T^{1,0}\check{D} \cap \bigoplus_{k=1}^{n} \mathrm{Hom}(\mathscr{F}^k/\mathscr{F}^{k+1}, \mathscr{F}^{k-1}/\mathscr{F}^k). \tag{9}$$

In [33], Schmid call a holomorphic mapping $\Psi : M \to \check{D}$ of a complex manifold M into $\check{D}$ horizontal if the tangent map $d\Psi : T^{1,0}M \to T^{1,0}\check{D}$ takes values in $T_h^{1,0}\check{D}$. The period map $\widetilde{\Phi} : \mathcal{T} \to D$ is horizontal due to Griffiths transversality.

Remark 1.1. We remark that elements in N_+ can be realized as nonsingular block lower triangular matrices with identity blocks in the diagonal; elements in B can be realized as nonsingular block upper triangular matrices. If $c, c' \in N_+$ such that $cB = c'B$ in $\check{D}$, then $c'^{-1}c \in N_+ \cap B = \{I\}$, i.e. $c = c'$. This means that the matrix representation in N_+ of the unipotent orbit $N_+(o)$ is unique. Therefore with a fixed base point $o \in \check{D}$, we can identify N_+ with its unipotent orbit $N_+(o)$ in $\check{D}$ by identifying an element $c \in N_+$ with $[c] = cB$ in $\check{D}$. Then $N_+ \subseteq \check{D}$ is meaningful. In particular, when the base point o is in D, we have $N_+ \cap D \subseteq D$.

Now we define

$$\check{\mathcal{T}} = \widetilde{\Phi}^{-1}(N_+ \cap D).$$

Then we show that $\mathcal{T}\backslash\check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}\backslash\check{\mathcal{T}}) \geq 1$.

Lemma 1.2. *Let $p \in \mathcal{T}$ be the reference point with $\widetilde{\Phi}(p) = \{F_p^n \subseteq F_p^{n-1} \subseteq \cdots \subseteq F_p^0\}$. Let $q \in \mathcal{T}$ be any point with $\widetilde{\Phi}(q) = \{F_q^n \subseteq F_q^{n-1} \subseteq \cdots \subseteq F_q^0\}$, then $\widetilde{\Phi}(q) \in N_+$ if and only if F_q^k is isomorphic to F_p^k for all $0 \leq k \leq n$.*

Proof. For any $q \in \mathcal{T}$, we choose an arbitrary adapted basis $\{\zeta_0, \cdots, \zeta_{m-1}\}$ for the given Hodge filtration $\{F_q^n \subseteq F_q^{n-1} \subseteq \cdots \subseteq F_q^0\}$. We fix $\{\eta_0, \cdots, \eta_{m-1}\}$ as the adapted basis for the Hodge filtration $\{F_p^n \subseteq F_p^{n-1} \subseteq \cdots \subseteq F_p^0\}$ at the base point p. Let $[A^{i,j}(q)]_{0 \leq i,j \leq n}$ be the transition matrix between the basis $\{\eta_0, \cdots, \eta_{m-1}\}$ and $\{\zeta_0, \cdots, \zeta_{m-1}\}$ for the same vector space $H_{\mathbb{C}}$, where $A^{i,j}(q)$ are the corresponding blocks. Then $\widetilde{\Phi}(q) \in N_+ = N_+B/B \subseteq \check{D}$ if and only if its matrix representation $[A^{i,j}(q)]_{0 \leq i,j \leq n}$ can be decomposed as $L(q) \cdot U(q)$, where $L(q)$ is a nonsingular block lower triangular matrix with identities in the diagonal blocks, and $U(q)$ is a nonsingular block upper triangular matrix. By basic linear algebra, we know that $[A^{i,j}(q)]$ has such decomposition if and only if $\det[A^{i,j}(q)]_{0 \leq i,j \leq k} \neq 0$ for any $0 \leq k \leq n$. In particular, we know that $[A(q)^{i,j}]_{0 \leq i,j \leq k}$ is the transition map between the bases of F_p^k and F_q^k. Therefore, $\det([A(q)^{i,j}]_{0 \leq i,j \leq k}) \neq 0$ if and only if F_q^k is isomorphic to F_p^k. $\square$

Proposition 1.3. *The subset $\check{\mathcal{T}}$ is an open complex submanifold in $\mathcal{T}$, and $\mathcal{T}\backslash\check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}\backslash\check{\mathcal{T}}) \geq 1$.*

Proof. From Lemma 1.2, one can see that $\check{D} \setminus N_+ \subseteq \check{D}$ is defined as an analytic subvariety by equations

$$\{q \in \check{D} : \det((A^{i,j}(q))_{0 \leq i,j \leq k}) = 0 \text{ for some } 0 \leq k \leq n\}.$$

Therefore N_+ is dense in $\check{D}$, and that $\check{D}\backslash N_+$ is an analytic subvariety, which is closed in $\check{D}$ and with $\mathrm{codim}_{\mathbb{C}}(\check{D}\backslash N_+) \geq 1$. We consider the period map $\widetilde{\Phi} : \mathcal{T} \to \check{D}$ as a holomorphic map to $\check{D}$, then $\mathcal{T} \setminus \check{\mathcal{T}} = \widetilde{\Phi}^{-1}(\check{D} \setminus N_+)$ is the preimage of $\check{D} \setminus N_+$ of the holomorphic map $\widetilde{\Phi}$. Therefore $\mathcal{T} \setminus \check{\mathcal{T}}$ is also an analytic subvariety and a closed set in $\mathcal{T}$. Because $\mathcal{T}$ is smooth and connected, $\mathcal{T}$ is irreducible. If $\dim_{\mathbb{C}}(\mathcal{T} \setminus \check{\mathcal{T}}) = \dim_{\mathbb{C}} \mathcal{T}$, then $\mathcal{T} \setminus \check{\mathcal{T}} = \mathcal{T}$ and $\check{\mathcal{T}} = \emptyset$, but this contradicts to the fact that the reference point p is in $\check{\mathcal{T}}$. Thus we conclude that $\dim_{\mathbb{C}}(\mathcal{T} \setminus \check{\mathcal{T}}) < \dim_{\mathbb{C}} \mathcal{T}$, and consequently $\mathrm{codim}_{\mathbb{C}}(\mathcal{T}\backslash\check{\mathcal{T}}) \geq 1$.

□

In the introduction, we have assumed that S admits a compactification $\overline{S}$ such that $\overline{S}$ is smooth and complete and $\overline{S} \setminus S$ is a divisor with simple normal crossings. Let $S' \supseteq S$ be the subset of $\overline{S}$ to which the period map $\Phi : S \to D/\Gamma$ extends continuously and $\Phi' : S' \to D/\Gamma$ be the extended map. Then one has the commutative diagram

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[rr, bend left, "\Phi"] & S' \arrow[r, "\Phi'"] & D/\Gamma. \end{tikzcd}$$

with $i : S \to S'$ the inclusion map.

Lemma 1.4. *S' is an open complex submanifold of $\overline{S}$ and the complex codimension of $\overline{S} \setminus S'$ is at least one.*

Proof. Here by abusing notation, in the following discussion we still denote by Γ the discrete subgroup in $\mathrm{Aut}(H_{\mathbb{Z}}, Q)$ containing the monodromy group which is the image of $\pi_1(S)$. As a standard procedure, after going to a finite cover we may assume Γ is torsion free, or even neat. See Theorem 3.6 in [26]. We will give two proofs of the lemma.

First we can use Theorem 9.6 of Griffiths in [11], see also Corollary 13.4.6 in [2], to get the Zariski open submanifold S'' of $\overline{S}$, where S'' contains all points of finite monodromy in $\overline{S}$, and hence of trivial monodromy in $\overline{S}$, since Γ is torsion free. Then the extension map $\Phi'' : S'' \to D/\Gamma$ is a proper holomorphic map. For the related discussion, see also page 705-706 in [38]. Since S'' is Zariski open in $\overline{S}$, we only need to prove that $S' = S''$. In fact, by the definition of S', we see that $S'' \subseteq S'$. Conversely, for any point $q \in S'$ with image $u = \Phi'(q) \in D/\Gamma$, we can choose the points $q_k \in S$, $k = 1, 2, \cdots$ such that $q_k \longrightarrow q$ with images $u_k = \Phi(q_k) \longrightarrow u$ as $k \longrightarrow \infty$. Since $\Phi'' : S'' \to D/\Gamma$ is proper, the sequence $\{q_k\}_{k=1}^{\infty} \subset (\Phi'')^{-1}(\{u_k\}_{k=1}^{\infty})$ has a limit point q in S'', that is to say $q \in S''$ and $S' \subseteq S''$. From this we see that $S' = S''$ is an open complex submanifold of $\overline{S}$ with $\mathrm{codim}_{\mathbb{C}}(\overline{S} \setminus S') \geq 1$.

For the second proof, note that $\overline{S}$ is smooth and $S \subseteq \overline{S}$ is Zariski open, we only need to show that S' is open in $\overline{S}$. To prove this we use the compactification space $\overline{D/\Gamma}$. There are several natural notions of the compactification space $\overline{D/\Gamma}$, see [19], [8, Page 2], [8, Page 29, 30], in which it is proved that the period map has continuous, even holomorphic extension. We can choose any one of them together with the continuous extension of the period map as

$$\overline{\Phi} : \overline{S} \to \overline{D/\Gamma}.$$

By the definition of S', $S' = \overline{\Phi}^{-1}(D/\Gamma)$. Since D/Γ is open and dense in the compacification $\overline{D/\Gamma}$, S' is therefore an open and dense submanifold of $\overline{S}$.

□

Let $\mathcal{T}'$ be the universal cover of S' with the universal covering map $\pi' : \mathcal{T}' \to S'$. We then have the following commutative diagram

$$\begin{tikzcd} \mathcal{T} \arrow[r, "i_{\mathcal{T}}"] \arrow[d, "\pi"] & \mathcal{T}' \arrow[r, "\widetilde{\Phi}'"] \arrow[d, "\pi'"] & D \arrow[d, "\pi_D"] \\ S \arrow[r, "i"] & S' \arrow[r, "\Phi'"] & D/\Gamma, \end{tikzcd} \tag{10}$$

where $i_{\mathcal{T}}$ is the lifting of $i \circ \pi$ with respect to the covering map $\pi' : \mathcal{T}' \to S'$ and $\widetilde{\Phi}'$ is the lifting of $\Phi' \circ \pi'$ with respect to the covering map $\pi_D : D \to D/\Gamma$. Then $\widetilde{\Phi}'$ is continuous. There are different choices of $i_{\mathcal{T}}$ and $\widetilde{\Phi}'$, but Lemma A.1 in the Appendix shows that we can choose $i_{\mathcal{T}}$ and $\widetilde{\Phi}'$ such that $\widetilde{\Phi} = \widetilde{\Phi}' \circ i_{\mathcal{T}}$. Let $\mathcal{T}_0 \subseteq \mathcal{T}'$ be defined by $\mathcal{T}_0 = i_{\mathcal{T}}(\mathcal{T})$.

Lemma 1.5. $\mathcal{T}_0 = \pi'^{-1}(S)$.

Proof. From the diagram (10), we see that $\pi'(\mathcal{T}_0) = \pi'(i_{\mathcal{T}}(\mathcal{T})) = i(\pi(\mathcal{T})) = S$, hence $\mathcal{T}_0 \subseteq \pi'^{-1}(S)$.

Conversely, for any $q \in \pi'^{-1}(S)$, we need to prove that $q \in \mathcal{T}_0$. Let $p = \pi'(q) \in S$. If there exists $r \in \pi^{-1}(p)$ such that $i_{\mathcal{T}}(r) = q$, then we are done. Otherwise, we can draw a curve γ from $i_{\mathcal{T}}(r)$ to q for some $r \in \pi^{-1}(p)$, as $\mathcal{T}'$ is connected and thus path connected. Then we get a circle $\Gamma = \pi'(\gamma)$ in S'. But Lemma A.2 in the Appendix implies that we can choose the circle Γ contained in S. Note that $p \in \Gamma$. Since $\pi : \mathcal{T} \to S$ is covering map, we can lift Γ to a unique curve $\widetilde{\gamma}$ from r to some $r' \in \pi^{-1}(p)$. Notice that both γ and $i_{\mathcal{T}}(\widetilde{\gamma})$ map to Γ via the the covering map $\pi' : \mathcal{T}' \to S'$, that is γ and $i_{\mathcal{T}}(\widetilde{\gamma})$ are both the lifts of Γ starting from the same point $i_{\mathcal{T}}(r)$. By the uniqueness of homotopy lifting, $i_{\mathcal{T}}(r') = q$, i.e. $q \in i_{\mathcal{T}}(\mathcal{T}) = \mathcal{T}_0$.

□

Lemma 1.5 implies that $\mathcal{T}_0$ is an open complex submanifold of $\mathcal{T}'$ and $\operatorname{codim}_{\mathbb{C}}(\mathcal{T}' \setminus \mathcal{T}_0) \geq 1$. Since $\widetilde{\Phi} = \widetilde{\Phi}' \circ i_{\mathcal{T}}$ is holomorphic, $\widetilde{\Phi}'|_{\mathcal{T}_0} : \mathcal{T}_0 \to D$ is also holomorphic. Since $\widetilde{\Phi}' : \mathcal{T}' \to D$ is continuous, $\widetilde{\Phi}'$ is locally bounded around $\mathcal{T}' \setminus \mathcal{T}_0$, then by applying Riemann extension theorem we have that $\widetilde{\Phi}' : \mathcal{T}' \to D$ is holomorphic.

Lemma 1.6. *The extended holomorphic map $\widetilde{\Phi}' : \mathcal{T}' \to D$ satisfies the Griffiths transversality.*

Proof. Let $\mathrm{T}^{1,0}_h D$ be the horizontal subbundle. Since $\widetilde{\Phi}' : \mathcal{T}' \to D$ is a holomorphic map, the tangent map

$$\widetilde{\Phi}'_* : \mathrm{T}^{1,0}\mathcal{T}' \to \mathrm{T}^{1,0}D$$

is at least continuous. We only need to show that the image of $\widetilde{\Phi}'_*$ is contained in the horizontal tangent bundle $\mathrm{T}^{1,0}_h D$.

Since $\mathrm{T}^{1,0}_h D$ is closed in $\mathrm{T}^{1,0}D$, $(\widetilde{\Phi}'_*)^{-1}(\mathrm{T}^{1,0}_h D)$ is closed in $\mathrm{T}^{1,0}\mathcal{T}'$. But $\widetilde{\Phi}'|_{\mathcal{T}_0}$ satisfies the Griffiths transversality, i.e. $(\widetilde{\Phi}'_*)^{-1}(\mathrm{T}^{1,0}_h D)$ contains $\mathrm{T}^{1,0}\mathcal{T}_0$, which is open in $\mathrm{T}^{1,0}\mathcal{T}'$. Hence $(\widetilde{\Phi}'_*)^{-1}(\mathrm{T}^{1,0}_h D)$ contains the closure of $\mathrm{T}^{1,0}\mathcal{T}_0$, which is $\mathrm{T}^{1,0}\mathcal{T}'$.

□

2. Further results of period domains from Lie theory

In this section, we study the structure of the Lie algebra $\mathfrak{g}$ by considering the root system, which is fundamental to our proof of Theorem 3.3 about the boundeness of the image of the restricted period map. Many results are from [14] and [33] to which we refer the reader for detailed proofs.

Now we define the Weil operator $\theta: \mathfrak{g} \to \mathfrak{g}$ by

$$\theta(X) = (-1)^k X, \text{ for } X \in \mathfrak{g}^{k,-k}.$$

Then θ is an involutive automorphism of $\mathfrak{g}$, and defined over $\mathbb{R}$. Let $\mathfrak{k}$ and $\mathfrak{p}$ be the $(+1)$ and (-1) eigenspaces of θ respectively. Considering the types, we have

$$\mathfrak{k} = \oplus_{k \text{ even}}\, \mathfrak{g}^{k,-k}, \ \mathfrak{p} = \oplus_{k \text{ odd}}\, \mathfrak{g}^{k,-k}.$$

Set

$$\mathfrak{k}_0 = \mathfrak{k} \cap \mathfrak{g}_0, \quad \mathfrak{p}_0 = \mathfrak{p} \cap \mathfrak{g}_0.$$

Then we have the decompositions

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad \mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$$

with the property that

$$[\mathfrak{k},\, \mathfrak{k}] \subseteq \mathfrak{k}, \quad [\mathfrak{p},\, \mathfrak{p}] \subseteq \mathfrak{k}, \quad [\mathfrak{k},\, \mathfrak{p}] \subseteq \mathfrak{p}.$$

Let $\mathfrak{g}_c = \mathfrak{k}_0 \oplus \sqrt{-1}\mathfrak{p}_0$. Then $\mathfrak{g}_c$ is also a real form of $\mathfrak{g}$. Let us denote the complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_c$ by τ_c, and the complex conjugation of $\mathfrak{g}$ with respect to the real form $\mathfrak{g}_0$ by τ_0.

Recall that on the complex Linear space $H_{\mathbb{C}}$ we can define an Hermitian inner product $(\cdot,\cdot)$ induced by the Poincaré bilinear form Q as

$$(u,v) = Q(Cu,\overline{v}) \ \ u,v \in H_{\mathbb{C}}, \tag{11}$$

where C is the Weil operator on $H_{\mathbb{C}}$ and defined over $\mathbb{R}$. Thus C can be considered as an element in $G_{\mathbb{R}}$, whose adjoint action on $\mathfrak{g}$ is just θ. For any $Z = X + \sqrt{-1}Y \in \mathfrak{g}_c$, where $X \in \mathfrak{k}_0$ and $Y \in \mathfrak{p}_0$, we have that $\forall\, u,v \in H_{\mathbb{C}}$

$$\begin{aligned}(Z\cdot u, v) &= Q(C((X+\sqrt{-1}Y)\cdot u),\overline{v}) \\ &= Q((X-\sqrt{-1}Y)\cdot Cu,\overline{v}) \\ &= -Q(Cu,(X-\sqrt{-1}Y)\cdot\overline{v}) \\ &= -Q(Cu,\overline{(X+\sqrt{-1}Y)\cdot v}) \\ &= -(u,Z\cdot v).\end{aligned} \tag{12}$$

Thus $\mathfrak{g}_c$ is the intersection of $\mathfrak{g}$ with the algebra of all skew Hermitian transforms with respect to the Hermitian inner product $(\cdot,\cdot)$. Using this result, Schmid in [33] proved that:

- $\mathfrak{g}_c$ is a compact real form of $\mathfrak{g}$, and the Killing form B

$$B(X,Y) = \mathrm{Trace}(\mathrm{ad}X \circ \mathrm{ad}Y), \quad X,Y \in \mathfrak{g}$$

restricts to a negative definite bilinear form $B|_{\mathfrak{g}_c}$ on $\mathfrak{g}_c$. Moreover, one has an Hermitian inner product $-B(\theta\ \cdot, \bar{\cdot})$ on $\mathfrak{g}$, making $\mathfrak{g}$ an Hermitian complex linear space.

Following this result, we have that $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$ is Cartan decomposition.

Now we define the Lie subgroup $G_c \subseteq G_{\mathbb{C}}$ corresponding to $\mathfrak{g}_c \subseteq \mathfrak{g}$. Then (12) implies that G_c contains the elements in $G_{\mathbb{C}}$ which preserve the Hermitian inner product $(\cdot, \cdot)$, i.e. G_c is the unitary subgroup of $G_{\mathbb{C}}$. Thus G_c is compact. As noted by Schmid, a compact real form in a connected complex semisimple Lie group is always connected and is its own normalizer, which implies that G_c is also connected.

The intersection

$$K = G_c \cap G_{\mathbb{R}}$$

is a compact subgroup of $G_{\mathbb{R}}$ with Lie algebra $\mathfrak{g}_c \cap \mathfrak{g}_0 = \mathfrak{k}_0$. In pages 278-279 of [33], Schmid showed that:

- K is a maximal compact subgroup of $G_{\mathbb{R}}$ and it meets every connected component of $G_{\mathbb{R}}$.
- $G_c \cap B = V$, which implies $V \subseteq K$ and their Lie algebras $\mathfrak{v}_0 \subseteq \mathfrak{k}_0$.

In [14], Griffiths and Schmid observed that:

- There exists a Cartan subalgebra $\mathfrak{h}_0$ of $\mathfrak{g}_0$ such that $\mathfrak{h}_0 \subseteq \mathfrak{v}_0 \subseteq \mathfrak{k}_0$ and $\mathfrak{h}_0$ is also a Cartan subalgebra of $\mathfrak{k}_0$;

Denote $\mathfrak{h}$ to be the complexification of $\mathfrak{h}_0$. Then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{k}$.

Now we review the root systems we need. Write $\mathfrak{h}_0^* = \mathrm{Hom}(\mathfrak{h}_0, \mathbb{R})$ and $\mathfrak{h}_{\mathbb{R}}^* = \sqrt{-1}\mathfrak{h}_0^*$. Then $\mathfrak{h}_{\mathbb{R}}^*$ can be identified with $\mathfrak{h}_{\mathbb{R}} := \sqrt{-1}\mathfrak{h}_0$ by the restricted Killing form $B|_{\mathfrak{h}_{\mathbb{R}}}$ on $\mathfrak{h}_{\mathbb{R}}$. Let $\rho \in \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$, one can define the following subspace of $\mathfrak{g}$,

$$\mathfrak{g}^{\rho} = \{x \in \mathfrak{g} | [h, x] = \rho(h)x \quad \text{for all } h \in \mathfrak{h}\}.$$

An element $\varphi \in \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$ is called a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$ if $\mathfrak{g}^{\varphi} \neq \{0\}$.

Let $\Delta \subseteq \mathfrak{h}_{\mathbb{R}}^* \simeq \mathfrak{h}_{\mathbb{R}}$ denote the space of nonzero $\mathfrak{h}$-roots. Then each root space

$$\mathfrak{g}^{\varphi} = \{x \in \mathfrak{g} | [h, x] = \varphi(h)x \text{ for all } h \in \mathfrak{h}\}$$

with respect to some $\varphi \in \Delta$ is one-dimensional over $\mathbb{C}$, generated by a root vector e_{φ}.

Since the involution θ is a Lie-algebra automorphism fixing $\mathfrak{k}$, we have

$$\theta([h, \theta(e_{\varphi})]) = [h, e_{\varphi}]$$

and hence

$$[h, \theta(e_{\varphi})] = \theta([h, e_{\varphi}]) = \varphi(h)\theta(e_{\varphi}),$$

for any $h \in \mathfrak{h}$ and $\varphi \in \Delta$. Thus $\theta(e_{\varphi})$ is also a root vector belonging to the root φ, so e_{φ} must be an eigenvector of θ. It follows that there is a decomposition of the roots Δ into the union $\Delta_{\mathfrak{k}} \cup \Delta_{\mathfrak{p}}$ of compact roots and non-compact roots with root spaces $\mathbb{C}e_{\varphi} \subseteq \mathfrak{k}$ and $\mathfrak{p}$ respectively.

The adjoint representation of $\mathfrak{h}$ on $\mathfrak{g}$ determins a decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\varphi \in \Delta} \mathfrak{g}^{\varphi}.$$

Let $\{\varphi_i : 1 \le i \le l\}$, $l = \mathrm{rank}(\mathfrak{g})$ be any simple root system of Δ, then we can introduce the notion of positivity in Δ, i.e. we can define Δ^+ to be the subset of Δ consisting of the roots in Δ which are positive integral linear combination of the simple roots. Let $\{h_i : 1 \le i \le l\}$ be the basis of $\mathfrak{h}_\mathbb{R}$ corresponding to the simple roots. That is to say that $\varphi_i(h) = B(h_i, h)$ for any $h \in \mathfrak{h}$, where $1 \le i \le l$ and the Cartan matrix $(B(h_i, h_j))$ is positive definite.

Following Serre [36], we can choose a Weyl basis $\{e'_\alpha : \alpha \in \Delta\}$ such that

$$
\begin{aligned}
&[\mathfrak{h}, \mathfrak{h}] = 0;\\
&[h, e'_\alpha] = \alpha(h) e'_\alpha, \ \forall h \in \mathfrak{h};\\
&[e'_\alpha, e'_{-\alpha}] = B(e'_\alpha, e'_{-\alpha}) h_\alpha, \ \forall \varphi \in \Delta;\\
&[e'_\alpha, e'_\beta] = N_{\alpha,\beta} e'_{\alpha+\beta}, \ \alpha + \beta \neq 0,
\end{aligned}
$$

where $N_{\alpha,\beta} = 0$ if $\alpha + \beta \neq 0$, $\alpha + \beta \notin \Delta$; $N_{\alpha,\beta} \neq 0$ if $\alpha + \beta \in \Delta$ with relation that

$$N_{-\alpha,-\beta} = -N_{\alpha,\beta} \in \mathbb{R}.$$

Now we define the real subspace of $\mathfrak{g}$ as

$$\mathfrak{g}'_c = \mathfrak{h}_0 + \sum_{\alpha\in\Delta} \mathbb{R}(e'_\alpha - e'_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}\sqrt{-1}(e'_\alpha + e'_{-\alpha}).$$

In fact, Theorem 6.3 in Chapter III of [16] shows that $\mathfrak{g}'_c$ is a compact real form of $\mathfrak{g}$ with complex conjugation τ'_c. Again by Theorem 7.1 in Chapter III of [16] and its proof, there exists one-parameter subgroup ϑ_t of automorphisms of $\mathfrak{g}$ such that $\vartheta_{\frac{1}{4}}(\mathfrak{g}'_c)$ is invariant under τ_c and $\vartheta_1 = (\tau_c \tau'_c)^2$. Then the proof of Theorem 7.2 and Corollary 7.3 in Chapter III of [16] show that $\mathfrak{g}_c = \vartheta_{\frac{1}{4}}(\mathfrak{g}'_c)$ and there exist some $X \in \mathfrak{g}$ such that $\vartheta_t = \exp(t \cdot \mathrm{ad}X)$. Since $\exp(\mathrm{ad}X) = (\tau_c\tau'_c)^2$ and $\tau_c|_{\mathfrak{h}_0} = \tau'_c|_{\mathfrak{h}_0} = \mathrm{id}$, X must be in $\mathfrak{h}$, hence $\vartheta_t|_{\mathfrak{h}_0} = \exp(t \cdot \mathrm{ad}X)|_{\mathfrak{h}_0} = \mathrm{id}$. Therefore we have proved that there exists an automorphism $\vartheta = \vartheta_{\frac{1}{4}}$ of $\mathfrak{g}$ such that $\mathfrak{g}_c = \vartheta(\mathfrak{g}'_c)$ and $\vartheta|_{\mathfrak{h}_0} = \mathrm{id}$.

For any $\alpha \in \triangle$, let $e_\alpha \triangleq \vartheta_{\frac{1}{4}}(e'_\alpha)$. Since $\vartheta_{\frac{1}{4}}$ is an automorphism, $\{e_\alpha : \alpha \in \triangle\}$ is also a Weyl basis. Then under the Weyl basis $\{e_\alpha : \alpha \in \triangle\}$,

$$
\begin{aligned}
\mathfrak{g}_c &= \vartheta_{\frac{1}{4}}(\mathfrak{g}'_c)\\
&= \mathfrak{h}_0 + \sum_{\alpha\in\Delta} \mathbb{R}(e_\alpha - e_{-\alpha}) + \sum_{\alpha\in\Delta} \mathbb{R}\sqrt{-1}(e_\alpha + e_{-\alpha})\\
&= \mathfrak{k}_0 + \sqrt{-1}\mathfrak{p}_0.
\end{aligned}
$$

Therefore, by considering types we have proved the following theorem.

Theorem 2.1. *There exist a basis $\{h_i : 1 \le i \le l\}$ of $\mathfrak{h}_\mathbb{R}$ and a Weyl basis $\{e_\varphi : \varphi \in \Delta\}$ such that*

$$
\begin{aligned}
&\tau_c(h_i) = \tau_0(h_i) = -h_i, &&\textit{for any } 1 \le i \le l;\\
&\tau_c(e_\varphi) = \tau_0(e_\varphi) = -e_{-\varphi}, &&\textit{for any } \varphi \in \Delta_\mathfrak{k};\\
&\tau_0(e_\varphi) = -\tau_c(e_\varphi) = e_{-\varphi}, &&\textit{for any } \varphi \in \Delta_\mathfrak{p},
\end{aligned}
\tag{13}
$$

and

$$\mathfrak{k}_0 = \mathfrak{h}_0 + \sum_{\varphi\in\Delta_{\mathfrak{k}}} \mathbb{R}(e_\varphi - e_{-\varphi}) + \sum_{\varphi\in\Delta_{\mathfrak{k}}} \mathbb{R}\sqrt{-1}(e_\varphi + e_{-\varphi}); \tag{14}$$

$$\mathfrak{p}_0 = \sum_{\varphi\in\Delta_{\mathfrak{p}}} \mathbb{R}(e_\varphi + e_{-\varphi}) + \sum_{\varphi\in\Delta_{\mathfrak{p}}} \mathbb{R}\sqrt{-1}(e_\varphi - e_{-\varphi}). \tag{15}$$

Lemma 2.2. *Let Δ be the set of $\mathfrak{h}$-roots as above. Then for each root $\varphi \in \Delta$, there is an integer $-n \le k \le n$ such that $e_\varphi \in \mathfrak{g}^{k,-k}$. In particular, if $e_\varphi \in \mathfrak{g}^{k,-k}$, then $e_{-\varphi} = -\tau_0(e_\varphi) \in \mathfrak{g}^{-k,k}$ for any $-n \le k \le n$.*

Proof. Let φ be a root, and e_φ be the generator of the root space $\mathfrak{g}^\varphi$, then $e_\varphi = \sum_{k=-n}^n e^{-k,k}$, where $e^{-k,k} \in \mathfrak{g}^{-k,k}$. Because $\mathfrak{h} \subseteq \mathfrak{v} \subseteq \mathfrak{g}^{0,0}$, the Lie bracket $[h, e^{-k,k}] \in \mathfrak{g}^{-k,k}$ for each k. Then the condition $[h, e_\varphi] = \varphi(h)e_\varphi$ implies that

$$\sum_{k=-n}^{n} [h, e^{-k,k}] = \sum_{k=-n}^{n} \varphi(h)e^{-k,k} \quad \text{for each } h \in \mathfrak{h}.$$

By comparing the type, we get

$$[h, e^{-k,k}] = \varphi(h)e^{-k,k} \quad \text{for each } h \in \mathfrak{h}.$$

Therefore $e^{-k,k} \in g^\varphi$ for each k. As $\{e^{-k,k}\}_{k=-n}^n$ forms a linear independent set, but g^φ is one dimensional, thus there is only one $-n \le k \le n$ with $e^{-k,k} \ne 0$.

□

Let us now introduce a lexicographic order (cf. pp.41 in [44] or pp.416 in [39]) in the real vector space $\mathfrak{h}_\mathbb{R}$ as follows: we fix an ordered basis $h_1, \cdots, h_l$ for $\mathfrak{h}_\mathbb{R}$. Then for any $h = \sum_{i=1}^l \lambda_i h_i \in \mathfrak{h}_\mathbb{R}$, we call $h > 0$ if the first nonzero coefficient is positive, that is, if $\lambda_1 = \cdots = \lambda_k = 0, \lambda_{k+1} > 0$ for some $1 \le k < l$. For any $h, h' \in \mathfrak{h}_\mathbb{R}$, we say $h > h'$ if $h - h' > 0$, $h < h'$ if $h - h' < 0$ and $h = h'$ if $h - h' = 0$.

In particular, let us identify the dual spaces $\mathfrak{h}_\mathbb{R}^*$ and $\mathfrak{h}_\mathbb{R}$, thus $\Delta \subseteq \mathfrak{h}_\mathbb{R}$. Let us choose a maximal linearly independent subset $\{\varphi_1, \cdots, \varphi_s\}$ of $\Delta_\mathfrak{p}$, then a maximal linearly independent subset $\{\varphi_{s+1}, \cdots, \varphi_l\}$ of $\Delta_\mathfrak{k}$. Then $\{\varphi_1, \cdots, \varphi_s, \varphi_{s+1}, \cdots, \varphi_l\}$ forms a basis for $\mathfrak{h}_\mathbb{R}^*$ since $\mathrm{Span}_\mathbb{R}\Delta = \mathfrak{h}_\mathbb{R}^*$. Then define the above lexicographic order in $\mathfrak{h}_\mathbb{R}^* \simeq \mathfrak{h}_\mathbb{R}$ using the ordered basis $\{\varphi_1, \cdots, \varphi_l\}$. In this way, we can also define

$$\Delta^+ = \{\varphi > 0 :\, \varphi \in \Delta\}; \quad \Delta_\mathfrak{p}^+ = \Delta^+ \cap \Delta_\mathfrak{p}.$$

Similarly we can define Δ^-, $\Delta_\mathfrak{p}^-$, $\Delta_\mathfrak{k}^+$, and $\Delta_\mathfrak{k}^-$.

Lemma 2.3. *Using the above notation, we have*

$$(\Delta_\mathfrak{k} + \Delta_\mathfrak{p}^\pm) \cap \Delta \subseteq \Delta_\mathfrak{p}^\pm; \tag{16}$$

$$(\Delta_\mathfrak{p}^+ + \Delta_\mathfrak{p}^+) \cap \Delta = \emptyset; \quad (\Delta_\mathfrak{p}^- + \Delta_\mathfrak{p}^-) \cap \Delta = \emptyset. \tag{17}$$

If one defines

$$\mathfrak{p}^\pm = \sum_{\varphi\in\Delta_\mathfrak{p}^\pm} \mathfrak{g}^\varphi \subseteq \mathfrak{p},$$

then $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, $[\mathfrak{p}^+, \mathfrak{p}^-] \subseteq \mathfrak{k}$ and

$$[\mathfrak{k}, \mathfrak{p}^\pm] \subseteq \mathfrak{p}^\pm; \tag{18}$$

$$[\mathfrak{p}^+,\ \mathfrak{p}^+] = 0,\ \ [\mathfrak{p}^-,\ \mathfrak{p}^-] = 0. \tag{19}$$

Proof. Relation (16) follows directly from the definition of lexicographic order, which also implies (18).

For (17), let $\varphi = \sum_{i>s} a_i\varphi_i,\ \psi = \sum_{i>s} b_i\varphi_i$ in $\Delta^+_{\mathfrak{p}}$, then $\varphi + \psi = \sum_{i>s}(a_i + b_i)\varphi_i \in \Delta^+_{\mathfrak{p}}$ and $e_\varphi,\ e_\psi,\ e_{\varphi+\psi} \in \mathfrak{p}$. Suppose that $\varphi + \psi \in \Delta$, then

$$0 \neq [e_\varphi,\ e_\psi] = N_{\varphi+\psi}e_{\varphi+\psi} \in [\mathfrak{p},\ \mathfrak{p}] \subseteq \mathfrak{k}.$$

Then $0 \neq e_{\varphi+\psi} \in \mathfrak{k} \cap \mathfrak{p} = \{0\}$, which is a contradiction. Therefore (17) holds, from which (19) follows.

□

Definition 2.4. *Two different roots $\varphi, \psi \in \Delta$ are said to be strongly orthogonal if and only if $\varphi \pm \psi \notin \Delta \cup \{0\}$, which is denoted by $\varphi \perp\!\!\!\perp \psi$.*

Lemma 2.5. *There exists a set of strongly orthogonal noncompact positive roots $\Lambda = \{\varphi_1, \cdots, \varphi_r\} \subseteq \Delta^+_{\mathfrak{p}}$ such that $0 < \varphi_1 < \cdots < \varphi_r$ and*

$$\mathfrak{A}_0 = \sum_{i=1}^{r} \mathbb{R}\left(e_{\varphi_i} + e_{-\varphi_i}\right)$$

is a maximal abelian subspace in $\mathfrak{p}_0$.

Proof. Let φ_1 be the minimum in $\Delta^+_{\mathfrak{p}}$, and φ_2 be the minimal element in $\{\varphi \in \Delta^+_{\mathfrak{p}} : \varphi \perp\!\!\!\perp \varphi_1\}$, then we obtain inductively an maximal ordered set of roots $\Lambda = \{\varphi_1, \cdots, \varphi_r\} \subseteq \Delta^+_{\mathfrak{p}}$, such that for each $1 \leq k \leq r$

$$\varphi_k = \min\{\varphi \in \Delta^+_{\mathfrak{p}} :\ \varphi \perp\!\!\!\perp \varphi_j \text{ for } 1 \leq j \leq k-1\}.$$

Because $\varphi_i \perp\!\!\!\perp \varphi_j$ for any $1 \leq i < j \leq r$, we have $[e_{\pm\varphi_i}, e_{\pm\varphi_j}] = 0$. Therefore

$$\mathfrak{A}_0 = \sum_{i=1}^{r} \mathbb{R}\left(e_{\varphi_i} + e_{-\varphi_i}\right)$$

is an abelian subspace of $\mathfrak{p}_0$.

We claim that $\varphi_1, \cdots, \varphi_r$ are $\mathbb{R}-$linearly independent. In fact, since $\varphi_i \pm \varphi_j \notin \Delta \cup \{0\}$ for $i \neq j$, we conclude that $B(\varphi_i, \varphi_j) = 0$, otherwise either $\frac{2B(\varphi_i,\varphi_j)}{B(\varphi_j,\varphi_j)}$ or $\frac{2B(\varphi_j,\varphi_i)}{B(\varphi_i,\varphi_i)}$ equals to ±1 and we assume that $\frac{2B(\varphi_i,\varphi_j)}{B(\varphi_j,\varphi_j)} = \pm1$, then the property of root system implies that

$$\varphi_i \mp \varphi_j = \varphi_i - \frac{2B(\varphi_i, \varphi_j)}{B(\varphi_j, \varphi_j)}\varphi_j \in \Delta,$$

which is a contradiction. Now we assume that $\sum_i k_i\varphi_i = 0$ with each $k_i \in \mathbb{R}$. Then

$$0 = B(\sum_i k_i\varphi_i, \sum_i k_i\varphi_i) = \sum_i k_i^2 B(\varphi_i, \varphi_i).$$

As $B|_{\mathfrak{h}_0}$ is negative definite, we conclude that each $k_i = 0$.

It remains to prove that α_0 is maximal. If it is not so, we can find $X \in \mathfrak{p}_0$ with

$$X = \sum_{\alpha\in\Delta^+_{\mathfrak{p}}\setminus\Lambda} \lambda_\alpha(e_\alpha + e_{-\alpha}) + \sum_{\alpha\in\Delta^+_{\mathfrak{p}}\setminus\Lambda} \mu_\alpha\sqrt{-1}(e_\alpha - e_{-\alpha}),\ \lambda_\alpha, \mu_\alpha \in \mathbb{R}$$

such that $[X, e_{\varphi_i} + e_{-\varphi_i}] = 0$ for $1 \leq i \leq r$. Let $c_\alpha = \lambda_\alpha + \sqrt{-1}\mu_\alpha$, then $X = \sum_{\alpha\in\Delta_\mathfrak{p}^+\setminus\Lambda}(c_\alpha e_\alpha + \overline{c_\alpha} e_{-\alpha})$ and for $1 \leq i \leq r$

$$\begin{aligned} 0 &= [X, e_{\varphi_i} + e_{-\varphi_i}] \\ &= \sum_{\alpha\in\Delta_\mathfrak{p}^+\setminus\Lambda} \big(c_\alpha(N_{\alpha+\varphi_i}e_{\alpha+\varphi_i} + N_{\alpha-\varphi_i}e_{\alpha-\varphi_i}) + \overline{c_\alpha}(N_{-\alpha+\varphi_i}e_{-\alpha+\varphi_i} + N_{-\alpha-\varphi_i}e_{-\alpha-\varphi_i})\big). \end{aligned} \tag{20}$$

By Lemma 2.3, for any $1 \leq i \leq r$, $\alpha + \varphi_i \notin \Delta$, hence $N_{\alpha+\varphi_i} = 0$. Similarly, $N_{-\alpha-\varphi_i} = 0$. Then (20) implies

$$\sum_{\alpha\in\Delta_\mathfrak{p}^+\setminus\Lambda} \big(c_\alpha N_{\alpha-\varphi_i}e_{\alpha-\varphi_i} + \overline{c_\alpha}N_{-\alpha+\varphi_i}e_{-\alpha+\varphi_i}\big) = 0. \tag{21}$$

For any $\alpha \in \Delta_\mathfrak{p}^+ \setminus \Lambda$, by the construction of Λ there exists an i such that $\alpha - \varphi_i$ lies in Δ, and then $N_{\alpha-\varphi_i} \neq 0$. Equation (21) then implies that $c_\alpha = 0$. Hence $X = 0$.

□

For further use, we also state a proposition about the maximal abelian subspace of $\mathfrak{p}_0$ as Lemma 6.3 in Chapter V of [16].

Proposition 2.6. *Let $\mathfrak{A}_0'$ be an arbitrary maximal abelian subspaces of $\mathfrak{p}_0$, then there exists an element $k \in K$ such that $\mathrm{Ad}(k) \cdot \mathfrak{A}_0 = \mathfrak{A}_0'$. Moreover, we have*

$$\mathfrak{p}_0 = \bigcup_{k\in K} \mathrm{Ad}(k) \cdot \mathfrak{A}_0,$$

where Ad *denotes the adjoint action of K on $\mathfrak{A}_0$.*

3. Boundedness of the period map

In the previous two sections, we have discussed the basic properties of the period domains. In particular, we have found the analytic subvariety $\mathcal{T}\setminus\check{\mathcal{T}}$ with $\mathrm{codim}_\mathbb{C}(\mathcal{T}\setminus\check{\mathcal{T}}) \geq 1$. In this section, we will prove the boundedness of the restricted period map $\widetilde{\Phi}|_{\check{\mathcal{T}}} : \check{\mathcal{T}} \to N_+ \cap D$ by using the structure theory of the complex semi-simple Lie algebra $\mathfrak{g}$.

Recall that we have fixed the reference points $p \in \mathcal{T}$ and $o = \widetilde{\Phi}(p) \in D$. Then N_+ can be viewed as a subset in $\check{D}$ by identifying it with its orbit $N_+(o)$ in $\check{D}$. At the base point $\widetilde{\Phi}(p) = o \in N_+ \cap D$, the tangent space $T_o^{1,0}N_+ = T_o^{1,0}D \simeq \mathfrak{n}_+$ and the exponential map $\exp : \mathfrak{n}_+ \to N_+$ is an isomorphism. Then the Hodge metric on $T_o^{1,0}D$ induces an Euclidean metric on N_+ so that $\exp : \mathfrak{n}_+ \to N_+$ is an isometry.

Also recall that we have defined $\check{\mathcal{T}} = \widetilde{\Phi}^{-1}(N_+ \cap D)$ and have shown that $\mathcal{T}\setminus\check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ with $\mathrm{codim}_\mathbb{C}(\mathcal{T}\setminus\check{\mathcal{T}}) \geq 1$. In this section, we prove that $\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$ is bounded in N_+ with respect to the Euclidean metric on N_+.

Let $\mathfrak{a} \subseteq \mathfrak{n}_+$ be the abelian subalgebra of $\mathfrak{n}_+$ determined by the tangent map of period map

$$\widetilde{\Phi}_* : \mathrm{T}^{1,0}\check{\mathcal{T}} \to \mathrm{T}^{1,0}D.$$

By Griffiths transversality, $\mathfrak{a} \subseteq \mathfrak{g}^{-1,1}$ is an abelian subspace. Let

$$A \triangleq \exp(\mathfrak{a}) \subseteq N_+$$

and $P : N_+ \cap D \to A \cap D$ be the projection map induced by the projection from N_+ to its subspace A. Then A, as a complex Euclidean space, has the induced Euclidean metric from N_+. We can consider A as an integral submanifold of the abelian algebra $\mathfrak{a}$ passing through the base point $o = \Phi(p)$, see page 248 in [3]. For the basic properties of integral manifolds of horizontal distribution, see Chapter 4 of [3], or [1] and [25].

The restricted period map $\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$ composed with the projection map P gives a holomorphic map

$$\Psi : \check{\mathcal{T}} \to A \cap D, \tag{22}$$

that is $\Psi = P \circ \widetilde{\Phi}|_{\check{\mathcal{T}}}$.

Lemma 3.1. *The image of the holomorphic map $\Psi : \check{\mathcal{T}} \to A \cap D$ is bounded in A with respect to the Euclidean metric on $A \subseteq N_+$.*

Proof. We need to show that there exists $0 \leq C < \infty$ such that for any $q \in \check{\mathcal{T}}$, $d_E(\Psi(p), \Psi(q)) \leq C$, where d_E is the Euclidean distance on A.

By the definition of A, for any $q \in \check{\mathcal{T}}$ there exists a unitary $X^+ \in \mathfrak{a}$ and a real number T_0 such that $\Psi(q) = \exp(T_0 X^+)$. Consider $X^- = \tau_0(X^+) \in \tau_0(\mathfrak{a})$, then $X = X^+ + X^- \in \mathfrak{a}_0 \triangleq \mathfrak{a} + \tau_0(\mathfrak{a}) \subseteq \mathfrak{p}_0$. For any $q \in \check{\mathcal{T}}$, there exists T such that $\gamma = \exp(tX) : [0, T] \to G_\mathbb{R}$ defines a geodesic from $\gamma(0) = I$ to $\gamma(T) \in G_\mathbb{R}$ such that $\gamma(T) \cdot o = \Psi(q) \in A \cap D$, where $\widetilde{\Phi}(p) = o$ is the base point.

Since $X \in \mathfrak{p}_0$, by Proposition 2.6, there exists $k \in K$ such that $X \in \mathrm{Ad}(k) \cdot \mathfrak{A}_0$. As the adjoint action of K on $\mathfrak{p}_0$ is unitary action and we are considering the length in this proof, we may simply assume that $X \in \mathfrak{A}_0$ up to a unitary transformation.

The following proof is an analogue of the proof of the Harish-Chandra embedding theorem for Hermitian symmetric spaces, see page 94 in [28].

Let $\Lambda = \{\varphi_1, \cdots, \varphi_r\} \subseteq \Delta^+_\mathfrak{p}$ be a set of strongly orthogonal roots given in Lemma 2.5. We denote $x_{\varphi_i} = e_{\varphi_i} + e_{-\varphi_i}$ and $y_{\varphi_i} = \sqrt{-1}(e_{\varphi_i} - e_{-\varphi_i})$ for any $\varphi_i \in \Lambda$. Then

$$\mathfrak{A}_0 = \mathbb{R}x_{\varphi_1} \oplus \cdots \oplus \mathbb{R}x_{\varphi_r}, \quad \text{and} \quad \mathfrak{A}_c = \mathbb{R}y_{\varphi_1} \oplus \cdots \oplus \mathbb{R}y_{\varphi_r},$$

are maximal abelian spaces in $\mathfrak{p}_0$ and $\sqrt{-1}\mathfrak{p}_0$ respectively. For any $X \in \mathfrak{A}_0$ there exists $\lambda_i \in \mathbb{R}$ for $1 \leq i \leq r$ such that

$$X = \lambda_1 x_{\varphi_1} + \lambda_2 x_{\varphi_2} + \cdots + \lambda_r x_{\varphi_r}.$$

Since $\mathfrak{A}_0$ is commutative, we have

$$\exp(tX) = \prod_{i=1}^{r} \exp(t\lambda_i x_{\varphi_i}).$$

Now for each $\varphi_i \in \Lambda$, we have $\mathrm{Span}_\mathbb{C}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{C})$ with

$$h_{\varphi_i} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad e_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{-\varphi_i} \mapsto \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix};$$

and $\mathrm{Span}_\mathbb{R}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\} \simeq \mathfrak{sl}_2(\mathbb{R})$ with

$$\sqrt{-1}h_{\varphi_i} \mapsto \begin{bmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{bmatrix}, \quad x_{\varphi_i} \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad y_{\varphi_i} \mapsto \begin{bmatrix} 0 & \sqrt{-1} \\ -\sqrt{-1} & 0 \end{bmatrix}.$$

Since $\Lambda = \{\varphi_1, \cdots, \varphi_r\}$ is a set of strongly orthogonal roots, we have that

$$\mathfrak{g}_{\mathbb{C}}(\Lambda) = \mathrm{Span}_{\mathbb{C}}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}\}_{i=1}^r \simeq (\mathfrak{sl}_2(\mathbb{C}))^r,$$
$$\text{and} \quad \mathfrak{g}_{\mathbb{R}}(\Lambda) = \mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\}_{i=1}^r \simeq (\mathfrak{sl}_2(\mathbb{R}))^r.$$

In fact, we know that for any $\varphi, \psi \in \Lambda$ with $\varphi \neq \psi$, $[e_{\pm\varphi}, e_{\pm\psi}] = 0$ since φ is strongly orthogonal to ψ; $[h_\varphi, h_\psi] = 0$, since $\mathfrak{h}$ is abelian; and

$$[h_\varphi, e_{\pm\psi}] = [[e_\varphi, e_{-\varphi}], e_{\pm\psi}] = -[[e_{-\varphi}, e_{\pm\psi}], e_\varphi] - [[e_{\pm\psi}, e_\varphi], e_{-\varphi}] = 0.$$

Then we denote $G_{\mathbb{C}}(\Lambda) = \exp(\mathfrak{g}_{\mathbb{C}}(\Lambda)) \simeq (SL_2(\mathbb{C}))^r$ and $G_{\mathbb{R}}(\Lambda) = \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda)) = (SL_2(\mathbb{R}))^r$, which are subgroups of $G_{\mathbb{C}}$ and $G_{\mathbb{R}}$ respectively. With the fixed reference point $o = \widetilde{\Phi}(p)$, we denote $D(\Lambda) = G_{\mathbb{R}}(\Lambda)(o)$ and $S(\Lambda) = G_{\mathbb{C}}(\Lambda)(o)$ to be the corresponding orbits of these two subgroups, respectively. Then we have the following isomorphisms,

$$D(\Lambda) = G_{\mathbb{R}}(\Lambda) \cdot B/B \simeq G_{\mathbb{R}}(\Lambda)/G_{\mathbb{R}}(\Lambda) \cap V, \tag{23}$$
$$S(\Lambda) \cap (N_+B/B) = (G_{\mathbb{C}}(\Lambda) \cap N_+) \cdot B/B \simeq G_{\mathbb{C}}(\Lambda) \cap N_+. \tag{24}$$

With the above notations, we will show that
(i) $D(\Lambda) \subseteq S(\Lambda) \cap (N_+B/B) \subseteq \check{D}$;
(ii) $D(\Lambda)$ is bounded inside $S(\Lambda) \cap (N_+B/B)$.

By Lemma 2.2, we know that for each pair of roots $\{e_{\varphi_i}, e_{-\varphi_i}\}$, there exists a positive integer k such that either $e_{\varphi_i} \in \mathfrak{g}^{-k,k} \subseteq \mathfrak{n}_+$ and $e_{-\varphi_i} \in \mathfrak{g}^{k,-k}$, or $e_{\varphi_i} \in \mathfrak{g}^{k,-k}$ and $e_{-\varphi_i} \in \mathfrak{g}^{-k,k} \subseteq \mathfrak{n}_+$. For notation simplicity, for each pair of root vectors $\{e_{\varphi_i}, e_{-\varphi_i}\}$, we may assume the one in $\mathfrak{g}^{-k,k} \subseteq \mathfrak{n}_+$ to be e_{φ_i} and denote the one in $\mathfrak{g}^{k,-k}$ by $e_{-\varphi_i}$. In this way, one can check that $\{\varphi_1, \cdots, \varphi_r\}$ may not be a set in $\Delta^+_{\mathfrak{p}}$, but it is a set of strongly orthogonal roots in $\Delta_{\mathfrak{p}}$.

Therefore, we have the following description of the above groups,

$$G_{\mathbb{R}}(\Lambda) = \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda)) = \exp(\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_1}, y_{\varphi_1}, \sqrt{-1}h_{\varphi_1}, \cdots, x_{\varphi_r}, y_{\varphi_r}, \sqrt{-1}h_{\varphi_r}\})$$
$$G_{\mathbb{R}}(\Lambda) \cap V = \exp(\mathfrak{g}_{\mathbb{R}}(\Lambda) \cap \mathfrak{v}_0) = \exp(\mathrm{Span}_{\mathbb{R}}\{\sqrt{-1}h_{\varphi_1}, \cdot, \sqrt{-1}h_{\varphi_r}\})$$
$$G_{\mathbb{C}}(\Lambda) \cap N_+ = \exp(\mathfrak{g}_{\mathbb{C}}(\Lambda) \cap \mathfrak{n}_+) = \exp(\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\}).$$

Thus by the isomorphisms in (23) and (24), we have

$$D(\Lambda) \simeq \prod_{i=1}^r \exp(\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\}) / \exp(\mathrm{Span}_{\mathbb{R}}\{\sqrt{-1}h_{\varphi_i}\},$$
$$S(\Lambda) \cap (N_+B/B) \simeq \prod_{i=1}^r \exp(\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_i}\}).$$

Let us denote $G_{\mathbb{C}}(\varphi_i) = \exp(\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_i}, e_{-\varphi_i}, h_{\varphi_i}) \simeq SL_2(\mathbb{C})$, $S(\varphi_i) = G_{\mathbb{C}}(\varphi_i)(o)$, and $G_{\mathbb{R}}(\varphi_i) = \exp(\mathrm{Span}_{\mathbb{R}}\{x_{\varphi_i}, y_{\varphi_i}, \sqrt{-1}h_{\varphi_i}\}) \simeq SL_2(\mathbb{R})$, $D(\varphi_i) = G_{\mathbb{R}}(\varphi_i)(o)$.

Now each point in $S(\varphi_i) \cap (N_+B/B)$ can be represented by

$$\exp(ze_{\varphi_i}) = \begin{bmatrix} 1 & z \\ 0 & 1 \end{bmatrix} \quad \text{for some } z \in \mathbb{C}.$$

Thus $S(\varphi_i) \cap (N_+B/B) \simeq \mathbb{C}$. In order to see $D(\varphi_i)$ in $G_{\mathbb{C}}/B$, we decompose each point in $D(\varphi_i)$ as follows. Let $z = a + bi$ for some $a, b \in \mathbb{R}$, then

$$\begin{aligned}
\exp(ax_{\varphi_i} + by_{\varphi_i}) &= \begin{bmatrix} \cosh|z| & \frac{z}{|z|}\sinh|z| \\ \frac{\overline{z}}{|z|}\sinh|z| & \cosh|z| \end{bmatrix} \\
&= \begin{bmatrix} 1 & \frac{z}{|z|}\tanh|z| \\ 0 & 1 \end{bmatrix} \begin{bmatrix} (\cosh|z|)^{-1} & 0 \\ 0 & \cosh|z| \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \frac{\overline{z}}{|z|}\tanh|z| & 1 \end{bmatrix} \\
&= \exp\left[(\frac{z}{|z|}\tanh|z|)e_{\varphi_i}\right] \exp\left[-\log(\cosh|z|)h_{\varphi_i}\right] \exp\left[(\frac{\overline{z}}{|z|}\tanh|z|)e_{-\varphi_i}\right] \\
&\equiv \exp\left[(\frac{z}{|z|}\tanh|z|)e_{\varphi_i}\right] \pmod B.
\end{aligned}$$

So elements of $D(\varphi_i)$ in $G_{\mathbb{C}}/B$ can be represented by $\exp[(z/|z|)(\tanh|z|)e_{\varphi_i}]$, i.e.

$$\begin{bmatrix} 1 & \frac{z}{|z|}\tanh|z| \\ 0 & 1 \end{bmatrix},$$

in which $\frac{z}{|z|}\tanh|z|$ is a point in the unit disc $\mathfrak{D}$ of the complex plane. Therefore the $D(\varphi_i)$ is a unit disc $\mathfrak{D}$ in the complex plane $S(\varphi_i) \cap (N_+B/B)$. Therefore

$$D(\Lambda) \simeq \mathfrak{D}^r \quad \text{and} \quad S(\Lambda) \cap N_+ \simeq \mathbb{C}^r.$$

So we have obtained both (i) and (ii). As a consequence, we get that for any $q \in \check{\mathcal{T}}$, $\Psi(q) \in D(\Lambda)$. This implies

$$d_E(\Psi(p), \Psi(q)) \leq \sqrt{r}$$

where d_E is the Eulidean distance on $S(\Lambda) \cap (N_+B/B)$.

To complete the proof, we only need to show that $S(\Lambda) \cap (N_+B/B)$ is totally geodesic in N_+B/B. In fact, the tangent space of N_+ at the base point is $\mathfrak{n}_+$ and the tangent space of $S(\Lambda) \cap N_+B/B$ at the base point is $\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\}$. Since $\mathrm{Span}_{\mathbb{C}}\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\}$ is a sub-Lie algebra of $\mathfrak{n}_+$, the corresponding orbit $S(\Lambda) \cap N_+B/B$ is totally geodesic in N_+B/B. Here the basis $\{e_{\varphi_1}, e_{\varphi_2}, \cdots, e_{\varphi_r}\}$ is an orthonormal basis with respect to the pull-back Euclidean metric.

□

As proved in Proposition 3 of Chapter 2 in [34], for the period map $\widetilde{\Phi} : \mathcal{T} \to D$, there is a stratification $\mathcal{T} = \cup_{\nu,k}\mathcal{T}_\nu^k$ such that if $\widetilde{\Phi}_\nu^k = \widetilde{\Phi}|_{\mathcal{T}_\nu^k}$, the rank of $d\widetilde{\Phi}_\nu^k$ is constant on $\mathcal{T}_\nu^k$. We will consider the image of a local neighborhood of $\mathcal{T}$ under the period map $\widetilde{\Phi}$ as a horizontal slice, which is given by the union of the image of each $\mathcal{T}_\nu^k$. Note that for the extended period map $\widetilde{\Phi}' : \mathcal{T}' \to D$, we can define horizontal slices in the same way, since the extended period map $\widetilde{\Phi}'$ still satisfies the Griffiths transversality.

Another point of view about horizontal slice is from complex analytic foliation. By using the natural Whitney stratification of the singular set of a complex analytic foliation, the following lemma is proved in [27] as Corollary 2.10.

Lemma 3.2. *Let E be a complex analytic singular foliation on a complex manifold M. Then there exists a family $\mathcal{L}$ of complex sub-manifolds of M such that $M = \cup_{L \in \mathcal{L}} L$ is disjoint union and that, for any $L \in \mathcal{L}$ and $p \in L$, we have $E(p) = T_pL$.*

Here E is considered as a coherent subsheaf of the tangent sheaf of M, and $E(p)$ denotes the tangent vectors in E at $p \in M$. We refer the reader to [27] for the details about singular complex analytic foliations. Note that in general the foliation E on the period domain D defined by the tangent map of the period map $\widetilde{\Phi}$ is a singular complex analytic foliation, since in this case we have $[E, E] = 0$.

With the notations above, each $L \in \mathcal{L}$ is identified with $\widetilde{\Phi}(\mathcal{T}_\nu^k)$ for some k and ν, and similarly for the images of $\widetilde{\Phi}'$. By Lemma 3.2, a neighborhood of any point in the image of the period map is a union of integral submanifolds of a singular foliation. We will also call these integral submanifolds the horizontal slices. One sees that the two descriptions of horizontal slices are equivalent.

The projection map $\pi : D \to G_{\mathbb{R}}/K$ is a one-to-one map when restricted to a horizontal slice by the Griffiths tranversality. We will need to use this fact in the following discussion.

Now let $\mathfrak{p}_+ = \mathfrak{p}/\mathfrak{p} \cap \mathfrak{b} \subseteq \mathfrak{n}_+$ denote the subspace of the holomorphic tangent space of D at the base point o, which is viewed as an Euclidean subspace of $\mathfrak{n}_+$, and consider the natural projection $P_+ : N_+ \cap D \to \exp(\mathfrak{p}_+) \cap D$ by viewing $\exp(\mathfrak{p}_+)$ as an Euclidean subspace of N_+. It is easy to see that the natural projection $\pi : D \to G_{\mathbb{R}}/K$, restricted to $\exp(\mathfrak{p}_+) \cap D$ is diffeomorphic to $G_{\mathbb{R}}/K$.

For simplicity we will assume that $A \cap D$ is contained in $\exp(\mathfrak{p}_+) \cap D$. Then there exists a neighborhood U of the base point o in D such that the image of $U \cap \widetilde{\Phi}(\check{\mathcal{T}})$ under the projection P_+ lies inside $A \cap D$, since they are both integral submanifolds of the horizontal distribution through the base point given by the abelian algebra $\mathfrak{a}$. From this we see that $P_+(\widetilde{\Phi}(\check{\mathcal{T}}))$ must lie inside $A \cap D$ completely, since $\widetilde{\Phi}(\check{\mathcal{T}})$ is a connected irreducible complex analytic variety, which is induced from the irreducibility of $\mathcal{T}$. Note that, with the above notations, $P : N_+ \cap D \to A \cap D$ may be considered as the composite of P_+ with the projection from $\exp(\mathrm{p}_+) \cap D$ to its subspace $A \cap D$.

Theorem 3.3. *The image of the restriction of the period map $\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$ is bounded in N_+ with respect to the Euclidean metric on N_+.*

Proof. In Lemma 3.1, we already proved that the image of $\Psi = P \circ \widetilde{\Phi}$ is bounded with respect to the Euclidean metric on $A \subseteq N_+$. Now together with the Griffiths transversality, we will deduce the boundedness of the image of $\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$ from the boundedness of the image of Ψ.

Since the projection $\pi : D \to G_{\mathbb{R}}/K$ restricted to $\exp(\mathfrak{p}_+) \cap D$ is a diffeomorphism, any two points in $N_+ \cap D$ are mapped to the same point in $\exp(\mathfrak{p}_+) \cap D$ via P_+, if and only if they are are mapped to the same point in $G_{\mathbb{R}}/K$ via π. Hence P_+ maps the fiber $\pi^{-1}(z') \cap N_+$ for $z' \in G_{\mathbb{R}}/K$ to a point $z \in \exp(\mathfrak{p}_+) \cap D$ with $\pi(z) = z'$. From this and the discussion above Theorem 3.3, we can see that for any $z \in \Psi(\check{\mathcal{T}})$, the restricted map $P|_{\widetilde{\Phi}(\check{\mathcal{T}})}$ is just the projection map from the intersection $\widetilde{\Phi}(\check{\mathcal{T}}) \cap \pi^{-1}(z')$ to z, where $z' = \pi(z) \in G_{\mathbb{R}}/K$.

With the above in mind, our proof can be divided into two steps. It is an elementary argument to apply the Griffiths transversality on $\mathcal{T}'$.

(i) We claim that there are only finite points in $(P|_{\widetilde{\Phi}(\check{\mathcal{T}})})^{-1}(z)$, for any $z \in \Psi(\check{\mathcal{T}})$.

Otherwise, we have $\{q_i\}_{i=1}^\infty \subseteq \check{\mathcal{T}}$ and $\{y_i = \widetilde{\Phi}(q_i)\}_{i=1}^\infty \subseteq (P|_{\widetilde{\Phi}(\check{\mathcal{T}})})^{-1}(z)$ with limiting point $y_\infty \in \pi^{-1}(z') \simeq K/V$, since K/V is compact. We project the points q_i to $q_i' \in S$ via

the universal covering map $p: \mathcal{T} \to S$. There must be infinite many q_i''s. Otherwise, we have a subsequence $\{q_{j_k}\}$ of $\{q_j\}$ such that $p(q_{j_k}) = q_{i_0}'$ for some i_0 and

$$y_{j_k} = \widetilde{\Phi}(q_{j_k}) = \gamma_k \widetilde{\Phi}(q_{j_0}) = \gamma_k y_{j_0},$$

where $\gamma_k \in \Gamma$ is the monodromy action. Since Γ is discrete, the subsequence $\{y_{j_k}\}$ is not convergent, which is a contradiction.

Now we project the points q_i on S via the universal covering map $p: \mathcal{T} \to S$ and still denote them by q_i without confusion. Then the sequence $\{q_i\}_{i=1}^{\infty} \subseteq S$ has a limiting point q_∞ in $\overline{S}$, where $\overline{S}$ is the compactification of S as defined in the introduction section. By continuity the period map $\Phi: S \to D/\Gamma$ can be extended over q_∞ with $\Phi(q_\infty) = \pi_D(y_\infty) \in D/\Gamma$, where $\pi_D: D \to D/\Gamma$ is the projection map. Thus q_∞ lies S'. Now we can regard the sequence $\{q_i\}_{i=1}^{\infty}$ as a convergent sequence in S' with limiting point $q_\infty \in S'$. We can also choose a sequence $\{\widetilde{q}_i\}_{i=1}^{\infty} \subseteq \mathcal{T}'$ with limiting point $\widetilde{q}_\infty \in \mathcal{T}'$ such that $\widetilde{q}_i$ maps to q_i via the universal covering map $\pi': \mathcal{T}' \to S'$ and $\widetilde{\Phi}'(\widetilde{q}_i) = y_i \in D$, for $i \geq 1$ and $i = \infty$. Since the extended period map $\widetilde{\Phi}': \mathcal{T}' \to D$ still satisfies the Griffiths transversality by Lemma 1.6, we can choose a small neighborhood U of $\widetilde{q}_\infty$ such that U, and thus the points $\widetilde{q}_i$ for i sufficiently large are mapped into a horizontal slice, which is a contradiction.

Denote $P|_{\widetilde{\Phi}'(\mathcal{T}')}$ to be the restricted map $P|_{\widetilde{\Phi}'(\mathcal{T}')}: N_+ \cap \widetilde{\Phi}'(\mathcal{T}') \to A \cap D$. In fact, a similar argument also implies that there are finite points in $(P|_{\widetilde{\Phi}'(\mathcal{T}')})^{-1}(z)$, for any $z \in \Psi(\check{\mathcal{T}})$. Furthermore, we have the following conclusions.

(ii) Let $r(z)$ be the cardinality of the fiber $(P|_{\widetilde{\Phi}'(\mathcal{T}')})^{-1}(z)$, for any $z \in \Psi(\check{\mathcal{T}})$. We claim that $r(z)$ is locally constant.

To be precise, for any $z \in \Psi(\check{\mathcal{T}})$, let $r = r(z)$ and choose points $x_1, \cdots, x_r$ in $N_+ \cap D$ such that $P|_{\widetilde{\Phi}'(\mathcal{T}')}(x_i) = z$. We can find a horizontal slice U_i around x_i such that

$$P|_{\widetilde{\Phi}'(\mathcal{T}')}: U_i \cap \widetilde{\Phi}'(\mathcal{T}') \to P|_{\widetilde{\Phi}'(\mathcal{T}')}(U_i) \tag{25}$$

is injective, $i = 1, \cdots, r$. We choose the balls B_i, $i = 1, \cdots, r$ small enough in $N_+ \cap D$ such that B_i's are mutually disjoint and $B_i \supseteq U_i$. We claim that there exists a small neighborhood $V \ni z$ in $A \cap D$ such that the restricted map

$$P|_{\widetilde{\Phi}'(\mathcal{T}')}: (P|_{\widetilde{\Phi}'(\mathcal{T}')})^{-1}(V) \cap B_i \to V \text{ is injective, for } i = 1, \cdots, r. \tag{26}$$

We define the pair of sequences (z, y) for some i as follows

($*_i$) $\{z_k\}_{k=1}^{\infty}$ is a convergent sequence in $A \cap D$ with limiting point z. $\{y_k\}_{k=1}^{\infty}$ is a convergent sequence in $N_+ \cap D$ with limiting point x_i such that $P(y_k) = z_k$ and $y_k \in B_i \setminus U_i$, for any $k \geq 1$.

We will prove that the pair of sequences (z, y) as ($*_i$) does not exist for any $i = 1, \cdots, r$, which implies that we can find a small neighborhood $V \ni z$ in $A \cap D$ such that

$$(P|_{\widetilde{\Phi}'(\mathcal{T}')})^{-1}(V) \cap B_i \subseteq U_i.$$

Hence (26) holds and $r(z)$ is locally constant.

In fact, if for some i there exists a pair of sequences (z, y) as ($*_i$), we can find a sequence $\{q_k\}_{k=1}^{\infty}$ in $\mathcal{T}'$ with limiting point $q_\infty \in \mathcal{T}'$ by a similar argument as (i) such that $\widetilde{\Phi}'(q_k) = y_k$ for any $k \geq 1$ and $\widetilde{\Phi}'(q_\infty) = x_i$. Since $\widetilde{\Phi}': \mathcal{T}' \to D$ still satisfies the Griffiths transversality due to Lemma 1.6, we can choose a small neighborhood $W \ni q_\infty$ in $\mathcal{T}'$ such

that $\widetilde{\Phi}'(W) \subseteq U_i$. Then for k sufficiently large, $y_k = \widetilde{\Phi}'(q_k) \in U_i$, which is a contradiction to $(*_i)$.

Therefore from (i) and (ii) we deduce that the image $\widetilde{\Phi}(\check{\mathcal{T}}) \subseteq N_+ \cap D$ is also bounded. □

Remark 3.4. The proof of Theorem 3.3 makes substantial use of the basic properties of the variation of Hodge structure arising from geometry, such as the integrability of the horizontal distribution and the local structure of the horizontal slices, as well as the quasi-projectivity of the family of polarized manifolds and the existence of the arithmetic discrete subgroup. Theorem 3.3 may not be valid for general horizontal holomorphic maps as pointed out by Professors Vincent Koziarz and Julien Maubon.

4. Proof of the Griffiths conjecture

In this section, we prove the Griffiths conjecture by using the boundedness of the restricted period map $\widetilde{\Phi}|_{\check{\mathcal{T}}} : \check{\mathcal{T}} \to N_+ \cap D$ and the Riemann extension theorem. This is our main result.

Theorem 4.1. (Main Theorem) *The image of $\widetilde{\Phi} : \mathcal{T} \to D$ lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+.*

Proof. According to Proposition 1.3, $\mathcal{T} \setminus \check{\mathcal{T}}$ is an analytic subvariety of $\mathcal{T}$ and the complex codimension of $\mathcal{T} \setminus \check{\mathcal{T}}$ is at least one; by Theorem 3.3, the holomorphic map $\widetilde{\Phi} : \check{\mathcal{T}} \to N_+ \cap D$ is bounded in N_+ with respect to the Euclidean metric. Thus by the Riemann extension theorem, there exists a holomorphic map $\widetilde{\Phi}_{\mathcal{T}} : \mathcal{T} \to N_+$, such that $\widetilde{\Phi}_{\mathcal{T}}|_{\check{\mathcal{T}}} = \widetilde{\Phi}|_{\check{\mathcal{T}}}$. Since as holomorphic maps, $\widetilde{\Phi}_{\mathcal{T}}$ and $\widetilde{\Phi}$ agree on the open subset $\check{\mathcal{T}}$, they must be the same on the entire $\mathcal{T}$. Therefore, the image of $\widetilde{\Phi}$ is in $N_+ \cap D$, and the image is bounded with respect to the Euclidean metric on N_+. As a consequence, we also get $\mathcal{T} = \check{\mathcal{T}} = \widetilde{\Phi}^{-1}(N_+ \cap D)$. □

From the proof of Theorem 3.3 and Theorem 4.1, we also have the following corollary, which improves the boundedness on $\mathcal{T}$ to its completion space $\mathcal{T}'$.

Corollary 4.2. *The image of the extended period map $\widetilde{\Phi}' : \mathcal{T}' \to D$ also lies in $N_+ \cap D$ and is bounded with respect to the Euclidean metric on N_+.*

Appendix A. Two topological lemmas

The two lemmas are elementary and may be well-known. We include their proofs here for the sake of completeness.

Lemma A.1. *There exists a suitable choice of $i_{\mathcal{T}}$ and $\widetilde{\Phi}'$ such that $\widetilde{\Phi}' \circ i_{\mathcal{T}} = \widetilde{\Phi}$.*

Proof. Recall the following commutative diagram as in (10)

$$\begin{tikzcd} \mathcal{T} \arrow[r, "i_{\mathcal{T}}"] \arrow[d, "\pi"] & \mathcal{T}' \arrow[r, "\widetilde{\Phi}'"] \arrow[d, "\pi'"] & D \arrow[d, "\pi_D"] \\ S \arrow[r, "i"] & S' \arrow[r, "\Phi'"] & D/\Gamma, \end{tikzcd} \tag{27}$$

Fix a reference point $p \in \mathcal{T}$. The relations $i \circ \pi = \pi' \circ i_{\mathcal{T}}$ and $\Phi' \circ \pi' = \pi_D \circ \widetilde{\Phi}'$ imply that $\pi_D \circ \widetilde{\Phi}' \circ i_{\mathcal{T}} = \Phi' \circ i \circ \pi = \Phi \circ \pi$. Therefore $\widetilde{\Phi}' \circ i_{\mathcal{T}}$ is a lifting map of Φ. On the other hand $\widetilde{\Phi} : \mathcal{T} \to D$ is also a lifting of Φ. In order to make $\widetilde{\Phi}' \circ i_{\mathcal{T}} = \widetilde{\Phi}$, one only needs to choose the suitable $i_{\mathcal{T}}$ and $\widetilde{\Phi}'$ such that these two maps agree on the reference point, i.e. $\widetilde{\Phi}' \circ i_{\mathcal{T}}(p) = \widetilde{\Phi}(p)$.

For an arbitrary choice of $i_{\mathcal{T}}$, we have $i_{\mathcal{T}}(p) \in \mathcal{T}'$ and $\pi'(i_{\mathcal{T}}(p)) = i(\pi(p))$. Considering the point $i_{\mathcal{T}}(p)$ as a reference point in $\mathcal{T}'$, we can choose $\widetilde{\Phi}'(i_{\mathcal{T}}(p))$ to be any point from $\pi_D^{-1}(\Phi'(i(\pi(p)))) = \pi_D^{-1}(\Phi(\pi(p)))$. Moreover the relation $\pi_D(\widetilde{\Phi}(p)) = \Phi(\pi(p))$ implies that $\widetilde{\Phi}(p) \in \pi_D^{-1}(\Phi(\pi(p)))$. Therefore we can choose $\widetilde{\Phi}'$ such that $\widetilde{\Phi}'(i_{\mathcal{T}}(p)) = \widetilde{\Phi}(p)$. With this choice, we have $\widetilde{\Phi}' \circ i_{\mathcal{T}} = \widetilde{\Phi}$. $\square$

Lemma A.2. *Let $\pi_1(S)$ and $\pi_1(S')$ be the fundamental groups of S and S' respectively, and suppose the group morphism*

$$i_* : \pi_1(S) \to \pi_1(S')$$

is induced by the inclusion $i : S \to S'$. Then i_* is surjective.

Proof. First notice that S and S' are both smooth manifolds, and $S \subseteq S'$ is open. Thus for each point $q \in S' \setminus S$ there is a disc $D_q \subseteq S'$ with $q \in D_q$. Then the union of these discs

$$\bigcup_{q \in S' \setminus S} D_q$$

forms a manifold with open cover $\{D_q : \ q \in \cup_q D_q\}$. Because both S and S' are second countable spaces, there is a countable subcover $\{D_i\}_{i=1}^{\infty}$ such that $S' = S \cup \bigcup\limits_{i=1}^{\infty} D_i$, where the D_i are open discs in S' for each i. Therefore, we have $\pi_1(D_i) = 0$ for all $i \geq 1$. Letting $S_k = S \cup \bigcup\limits_{i=1}^{k} D_i$, we get

$$\pi_1(S_k) * \pi_1(D_{k+1}) = \pi_1(S_k) = \pi_1(S_{k-1} \cup D_k), \quad \text{ for any } k.$$

We know that $\operatorname{codim}_{\mathbb{C}}(S' \backslash S) \geq 1$. Therefore since $D_{k+1} \backslash S_k \subseteq D_{k+1} \backslash S$, we have

$$\operatorname{codim}_{\mathbb{C}}(D_{k+1} \backslash S_k) \geq 1$$

for any k. As a consequence we can conclude that $D_{k+1} \cap S_k$ is path-connected. Hence we can apply the Van Kampen Theorem on $S_k = D_{k+1} \cup S_k$ to conclude that for every k, the following group homomorphism is surjective:

$$\pi_1(S_k) = \pi_1(S_k) * \pi_1(D_{k+1}) \twoheadrightarrow \pi_1(S_k \cup D_{k+1}) = \pi_1(S_{k+1}).$$

Thus we get the directed system:

$$\pi_1(S) \twoheadrightarrow \pi_1(S_1) \twoheadrightarrow \cdots\cdots \twoheadrightarrow \pi_1(S_k) \twoheadrightarrow \cdots\cdots$$

By taking the direct limit of this directed system, we get the surjectivity of the group homomorphism $\pi_1(S) \to \pi_1(S')$.

$\square$

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